DFTT 10/95

Quantum Hall Fluids as $W_{1+\infty}$ Minimal Models*

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Abstract

We review our recent work on the algebraic characterization of quantum Hall fluids. Specifically, we explain how the incompressible quantum fluid ground states can be classified by effective edge field theories with the $W_{1+\infty}$ dynamical symmetry of "quantum area-preserving diffeomorphisms". Using the representation theory of $W_{1+\infty}$, we show how all fluids with filling factors $\nu = m/(pm+1)$ and $\nu = m/(pm-1)$ with m and p positive integers, p even, correspond exactly to the $W_{1+\infty}$ *minimal models*.

February 1995

*To appear in the proceedings of the conference on *Statistical Mechanics and Quantum Field Theory*, University of Southern California, Los Angeles (U.S.A.), May 1994 and of the summer school on *Particles, Fields and Strings*, Banff (Canada), August 1994.

†Supported by a Profil 2 fellowship of the Swiss National Science Foundation.

1 Introduction

The quantum Hall effect [1] provides fascinating examples of *quantum fluids*. At low temperatures, interacting planar electrons in high magnetic fields B have strong quantum correlations which lead to collective motion and macroscopic quantum effects. These find their experimental evidence in a discrete series of plateaus at rational values of the Hall conductivity:

$$\begin{aligned} \sigma_{xy} &= \frac{e^2}{h}\,\nu \ , \\ \nu &= 1, \frac{1}{3}, \frac{1}{5}, \frac{2}{7}, \dots, 2, \dots . \end{aligned} \tag{1.1}$$

Corresponding to these plateaus the longitudinal conductivity σ_{xx} vanishes. The same plateaus are observed in several materials, signalling *universality*. Another experimental result is the remarkable *exactness* of these rational values of ν; the experimental error is $\Delta\nu = 10^{-8}$ for integer ν.

The current understanding of the quantum Hall effect is based on the seminal work of Laughlin [2]. The main idea is the existence of *incompressible quantum fluids* at specific rational values $\bar{\rho} = \nu B/2\pi$ ($\hbar = 1$, $c = 1$) of the electron density. These are very stable, macroscopic quantum states with uniform density and an energy gap. Incompressibility accounts for the lack of low-lying conduction modes, which causes σ_{xx} to vanish, while the Hall conduction is realized as an overall rigid motion of the uniform droplet, which gives eq. (1.1).

While Laughlin's theory is very successful, the observed exactness and universality calls for a *fundamental principle* underlying it. Indeed, the *universality* observed in experiments calls for an effective theory approach at long distances, while the extreme precision of the rational values of ν suggests that dynamics is constrained by *symmetry*. Both facts suggest an analogy with two-dimensional critical phenomena, which are classified by conformal field theories [3].

The effective field theory approach, developed by Landau, Ginsburg, Wilson and others [4], does not attempt to solve the microscopic many-body dynamics, but rather it guesses the macroscopic physics generated by this dynamics. The variables of the effective field theory are the relevant low-energy (long-distance) degrees of freedom, which are characterized by a specific symmetry. They describe universal properties, which are independent of the microscopic details. This approach is well suited for the quantum Hall effect, given the very precise and universal values of the Hall conductivity.

In the following, we shall present an overview of our [5][6][8][7][9][10][11] and related [12][13] recent work on the effective field theory approach to the quantum Hall effect. This is based on an *algebraic characterization* of incompressible quantum fluids.

2 Dynamical symmetry and kinematics of incompressible fluids

In this section, we shall review the dynamical symmetry characterizing *chiral*, two-dimensional, incompressible quantum fluids and indicate how this leads uniquely to the construction of the Hilbert spaces of low-energy excitations.

Classical fluids

A classical incompressible fluid is defined by its distribution function

$$\rho(z,\bar{z},t) = \rho_0\ \chi_{S_A(t)}\ , \qquad \rho_0 \equiv \frac{N}{A}\ , \tag{2.1}$$

where $\chi_{S_A(t)}$ is the characteristic function for a surface $S_A(t)$ of area A, and $z = x+iy$, $\bar{z} = x-iy$ are complex coordinates on the plane. Since the particle number N and the average density ρ_0 are constant, the area A is also *constant*. The only possible change in response to external forces is in the shape of the surface. The shape changes at constant area can be generated by *area-preserving diffeomorphisms* of the two-dimensional plane. Thus, the configuration space of a classical incompressible fluid can be generated by applying these transformations to a reference droplet.

Next we recall the Liouville theorem, which states that canonical transformations preserve the phase-space volume. Area-preserving diffeomorphisms are, therefore, canonical transformations of a two-dimensional phase space. In order to use the formalism of canonical transformations, we treat the original coordinate plane as a *phase space*, by postulating non-vanishing Poisson brackets between z and $\bar{z}$. We do this by defining the dimensionless Poisson brackets

$$\{f,g\} \equiv \frac{i}{\rho_0}\ \left(\partial f \bar{\partial} g - \bar{\partial} f \partial g\right)\ , \tag{2.2}$$

where $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial\bar{z}$, so that

$$\{z,\bar{z}\} = \frac{i}{\rho_0}\ . \tag{2.3}$$

Note that the Poisson brackets select a preferred *chirality*, because they are not invariant under the two-dimensional parity transformation $z \longrightarrow \bar{z}, \ \bar{z} \longrightarrow z$; in the quantum Hall effect, the parity breaking is due to the external magnetic field.

Area-preserving diffeomorphisms, *i.e.*, canonical transformations, are usually defined in terms of a generating function $\mathcal{L}(z,\bar{z})$ of both "coordinate" and "momentum", as follows:

$$\delta z \ = \ \{\mathcal{L}, z\} \ , \qquad \delta \bar{z} \ = \ \{\mathcal{L}, \bar{z}\} \ . \tag{2.4}$$

A basis of (dimensionless) generators is given by

$$\mathcal{L}^{(cl)}_{n,m} \equiv \rho_0^{\frac{n+m}{2}} \ z^n \bar{z}^m \ . \tag{2.5}$$

These satisfy the classical w_∞ algebra [14]

$$\left\{ \mathcal{L}^{(cl)}_{n,m}, \mathcal{L}^{(cl)}_{k,l} \right\} = -i \ (mk - nl) \ \mathcal{L}^{(cl)}_{n+k-1,m+l-1} \ . \tag{2.6}$$

Let us now discuss how w_∞ transformations can be used to generate the configuration space of classical excitations above the ground state. These configurations have a classical energy due to the inter-particle interaction and the external confining potential, whose specific form is not needed here. Let us assume a generic convex and rotation-invariant energy function, such that the minimal energy configuration ρ_{GS} has the shape of a disk of radius R:

$$\rho_{GS}(z,\bar{z}) = \rho_0 \ \Theta\left(R^2 - z\bar{z} \right) \ , \tag{2.7}$$

where Θ is the Heaviside step function. The classical "small excitations" around this ground state configuration are given by the infinitesimal deformations of ρ_{GS} under area-preserving diffeomorphisms,

$$\delta\rho_{n,m} \equiv \left\{ \mathcal{L}^{(cl)}_{n,m}, \rho_{GS} \right\} \ . \tag{2.8}$$

Using the Poisson brackets (2.2) , we obtain

$$\delta\rho_{n,m} = i \left(\rho_0 R^2 \right)^{\frac{n+m}{2}} (m-n) \ \mathrm{e}^{i(n-m)\theta} \ \delta\left(R^2 - z\bar{z} \right) \ . \tag{2.9}$$

These correspond to density fluctuations localized on the sharp boundary (which is parametrized by the angle θ) of the classical droplet. Due to the dynamics provided by the energy function, they will propagate on the boundary with a frequency ω_k dependent on the angular momentum $k \equiv (n-m)$, thereby turning into *edge waves*. These are the eigenoscillations of the classical incompressible fluid.

Another type of excitations are classical vortices in the bulk of the droplet, which correspond to localized holes or dips in the density. The absence of density waves, due to incompressibility, implies that any localized density excess or defect is transmitted completely to the boundary, where it is seen as a further edge deformation. For each given vorticity in the bulk, we can then construct the corresponding basis of edge waves in a fashion analogous to (2.8) . Thus, the configuration space of the excitations of a classical incompressible fluid (of a given vorticity) is spanned by infinitesimal w_∞ transformations. This is the *dynamical symmetry* of classical incompressible fluids.

Quantum fluids and their edge excitations

The quantum [‡] version of the chiral, incompressible fluids is given by the Laughlin theory of the plateaus of the quantum Hall effect [2] . The simplest example of such a macroscopic quantum state is a fully filled Landau level (filling fraction $\nu = 1$). Generically, it possesses three types of excitations. First, there are *gapless edge excitations* [15] , which are the quantum descendants of the classical edge waves described before. These are particle-hole excitations across the Fermi surface represented by the edge of the droplet of radius R [16]. They are *gapless* because their energy, of $O(1/R)$, vanishes for $R \to \infty$. Second, there are localized quasi-particle and quasi-hole excitations, which have a finite gap. These are the quantum analogs of the classical vortices and correspond to the anyon excitations [2] with fractional charge, spin and statistics [17]. As in the classical case, they manifest themselves as charged excitations at the edge, owing to incompressibility. The third type of excitations are two-dimensional density waves in the bulk, the magnetoplasmons and (for $\nu < 1$) the magnetophonons [18]. These have higher gaps and are not included in our effective field theory approach.

In the previous section, we have explained the connection between the classical edge waves and the generators of the algebra w_∞ of area-preserving diffeomorphisms. In the quantum theory, there is a corresponding relation between edge excitations and the generators of the quantum version of w_∞, called $W_{1+\infty}$ [14] . This algebra is obtained by replacing the Poisson brackets refpob with quantum commutators: $i\{\ ,\ \} \to [\ ,\ ]$, and by taking the thermodynamic limit [6] .

In this limit, the radius of the droplet grows as $R \propto \ell\sqrt{N}$, where $\ell = \sqrt{2/(eB)}$ is the magnetic length and B the magnetic field. Quantum edge excitations, instead, are confined to a boundary annulus of finite size $O(\ell)$. In the $N \to \infty$ limit, therefore, edge excitations become the particle hole excitations of a relativistic theory describing

‡ Throughout this paper we shall use units such that $c = 1$, $\hbar = 1$.

a Weyl (chiral) fermion living on the one-dimensional edge of the droplet [6] . In this limit, the quantum incompressible fluid becomes the Dirac sea for this relativistic theory. Charged fermions represent instead quasi-particle excitations.

The field operator for the Weyl fermion [§] is given by [3]

$$F_R(\theta) \;=\; \frac{1}{\sqrt{R}} \sum_{k=-\infty}^{\infty} \mathrm{e}^{i(k-1/2)\theta}\; b_k \;, \qquad (|z|=R\;,\; t=0)\;, \tag{2.10}$$

where θ parametrizes the circular boundary, b_k and $b_k^\dagger$ are fermionic Fock space operators satisfying $\left\{ b_l, b_k^\dagger \right\} = \delta_{l,k}$, and k is the angular momentum measured with respect to the ground state value.

The generators of the quantum algebra $W_{1+\infty}$ are represented in this Fock space by the bilinears

$$\begin{aligned} V_n^j \;&=\; \int_0^{2\pi} \frac{d\theta}{2\pi}\; :\; F^\dagger(\theta)\; \mathrm{e}^{-in\theta}\; g_n^j\,(i\partial_\theta)\;\; F(\theta)\;\; : \\ &=\; \sum_{k=-\infty}^{\infty} p(k,n,j)\;\; : b_{k-n}^\dagger\; b_k : \;, \qquad j \ge 0\;. \end{aligned} \tag{2.11}$$

In this expression, $F(\theta) = F_R(\theta)\; \mathrm{e}^{i\theta/2}\sqrt{R}$ is the canonical form of the Weyl field operator of conformal field theory. The factor $g_n^j\,(i\partial_\theta)$ is a j-th order polynomial in $i\partial_\theta$, whose form specifies the basis of operators and guarantees the hermiticity $(V_n^j)^\dagger = V_{-n}^j$. The coefficients $p(k,n,j)$ are also j-th order polynomials in k which we do not need to specify here (see [10]). The $W_{1+\infty}$ algebra reads

$$\left[\; V_n^i, V_m^j \;\right] = (jn - im)\; V_{n+m}^{i+j-1} + q(i,j,m,n)\; V_{n+m}^{i+j-3} + \ldots + c^i(n)\; \delta^{i,j}\delta_{n+m,0}\;. \tag{2.12}$$

Here, $i+1 = h \ge 1$ represents the "conformal spin" of the generator V_n^i, while $-\infty < n < +\infty$ is the angular momentum (the Fourier mode on the circle). The first term on the right-hand-side of (2.12) reproduces the classical w_∞ algebra (2.6) by the correspondence $\mathcal{L}_{i-n,i}^{(cl)} \to V_n^i$ and identifies $W_{1+\infty}$ as the algebra of "quantum area-preserving diffeomorphisms". The additional terms are quantum operator corrections with polynomial coefficients $q(i,j,n,m)$, due to the algebra of higher derivatives [14]. Moreover, the c-number term $c^i(n)$ is the quantum *anomaly*, a relativistic effect due to the renormalization of operators acting on the infinite Dirac sea. It is diagonal in the spin indices for our choice of basis for the g_k^i (see [10]). Finally, the normal ordering (: :) of the Fock operators takes care of the renormalization [3] .

Let us analyse the generators V_n^0 and V_n^1 of lowest conformal spin. From (2.11) we see that the V_n^0 are Fourier modes of the fermion density evaluated at the edge

[§] Hereafter, we choose units such that $\ell = 1$.

$|z| = R$; thus, V_0^0 measures the edge charge. Instead, the V_n^1 are vector fields which generate angular momentum transformations on the edge, such that V_0^1 measures the angular momentum of edge excitations. Their algebra is given by

$$\left[\ V_n^0, V_m^0\ \right] = c\ n\ \delta_{n+m,0} \qquad , \tag{2.13}$$

and

$$\begin{aligned} \left[\ V_n^1, V_m^0\ \right] &= -m\ V_{n+m}^0\ , \\ \left[\ V_n^1, V_m^1\ \right] &= (n-m)\ V_{n+m}^1 + \frac{c}{12}\left(n^3 - n\right)\delta_{n+m,0}\ , \end{aligned} \tag{2.14}$$

with $c = 1$. These equations show that the V_n^0 and V_n^1 operators satisfy the Abelian current (Kac-Moody) algebra $\widehat{U(1)}$ and the Virasoro algebra, respectively [3]. For unitary $W_{1+\infty}$ theories, the central charge c can be any *positive integer* [19] [20] .

Following the standard procedure of two-dimensional conformal field theory, we define a $W_{1+\infty}$ theory as the Hilbert space given by a set of irreducible, highest-weight representations of the $W_{1+\infty}$ algebra, closed under the *fusion rules* for making composite excitations [3] . Any representation contains an infinite number of states, corresponding to all the particle-hole excitations above a bottom state, the so-called *highest weight* state. This can be, for example, the ground-state $|\Omega\rangle$ corresponding to the incompressible quantum fluid. The particle-hole excitations can be written as

$$V_{-n_1}^{i_1}\ V_{-n_2}^{i_2} \cdots V_{-n_s}^{i_s} |\ \Omega\ \rangle\ , \quad n_1 \geq n_2 \geq \cdots \geq n_s > 0\ , \quad i_1, \ldots, i_s \geq 0\ , \tag{2.15}$$

while the positive modes ($n_i < 0$) annihilate $|\ \Omega\ \rangle$. Here $k = \sum_j n_j$ is the total angular momentum of the edge excitation. The number of independent states at fixed k, i.e. the number of independent V_n^i current modes, is finite; its actual value depends on the type of representation and the central charge.

Furthermore, any charged edge excitation, together with its tower of particle-hole excitations, also forms an irreducible, highest-weight representation of $W_{1+\infty}$. The states in this representation have the same form of (2.15), but the bottom state $|Q\rangle$ now represents a quasi-particle inside the droplet. The charge Q and the spin J of the quasi-particle are given by the eigenvalues of the operators¶ V_0^0 and V_0^1, respectively:

$$V_0^0|Q\rangle = Q|Q\rangle\ ,\ V_0^1|Q\rangle = J|Q\rangle\ . \tag{2.16}$$

Moreover, the statistics θ/π of quasi-particles is equal to twice the spin J. For $W_{1+\infty}$ representations , all the operators V_0^i are simultaneously diagonal and assign other

¶ More precisely, V_0^0 measures the charge spilled to the edge, which is minus the charge of the quasi-particle in the bulk, due to overall charge conservation.

quantum numbers to the quasi-particle, $V_0^i|Q\rangle = m_i(Q)|Q\rangle$, $i \geq 2$, which are known polynomials in the charge Q [19]. These quantum numbers measure the radial moments of the charge distribution of a quasi-particle (see [10]); their fixed functional form indicates the rigidity of density modulations of the quantum incompressible fluid.

Classification of QHE universality classes

Besides the explicit example for $\nu = 1$, leading to a theory with $c = 1$, it has been shown in general that the algebra (2.12) is the unique quantization of the w_∞ algebra in the $(1+1)$-dimensional field theory on the circle [21] . We shall therefore characterize quantum incompressible fluids as $W_{1+\infty}$ theories [9].

This characterization provides a powerful classification scheme for quantum Hall universality classes. These can in fact be classified by using the recently developed representation theory [19] [20] of $W_{1+\infty}$. We shall classify quantum Hall universality classes by the following *kinematical data*:

i) the quantum numbers V_0^i of quasi-particle excitations, in particular their charge and fractional statistics;

ii) the number of particle-hole excitations of given angular momentum, i.e. the *degeneracies* of states on top of the ground state (2.15);

iii) the Hall conductivity $\sigma_H = (e^2/h)\nu$, which is proportional to the filling fraction ν of the ground state.

The Hall current produced by an external electric field is actually given by the *chiral anomaly* of the (1+1)-dimensional edge theory [6].

3 Existing theories of edge excitations and experiments

Before developing this classification program, we would like to briefly review the existing theories of the quantum Hall effect and discuss their description of the experimental data.

Hierarchical trial wave functions

The Laughlin theory of the incompressible fluid [2] was originally developed for the Hall conductivities $\sigma_{xy} = (e^2/h)\nu$, where $\nu = 1, 1/3, 1/5, 1/7, \ldots$ are the filling fractions. Afterwards, a hierarchical generalization of these trial wave functions was introduced by Haldane and Halperin [22], in order to describe other observed filling fractions. Therefore, by the *hierarchy* problem we usually mean the classification of stable ground states (and their excitations) corresponding to all observed plateaus. Naturally, the stability is related to the order of iteration of the hierarchical construction, starting from the integer fillings, then the Laughlin fillings and so forth. The Haldane-Halperin hierarchy is not completely satisfactory, because it produces ground states for too many filling fractions, already at low order of iteration. On the contrary, the experiments show only some stable ground states (see fig. 1). Although numerical experiments show that the hierarchical wave functions are rather accurate, their construction lacks a good control of stability.

Another hierarchical construction of wave functions, which match most of the experimental plateaus to lowest order of the hierarchy, has been proposed by Jain [23]. Jain abstracted from Laughlin's work the concept of *composite fermion*, a local bound state of the electron and an even number of flux quanta. Due to yet unknown dynamical reasons, the composite fermions are stable quasi-particles, which interact weakly among themselves. Moreover, the strongly-interacting electrons at fractional filling can be mapped into composite fermions at effective integer filling. Therefore, the stability of the observed ground states with fractional filling can be related to the stability of completely filled Landau levels. The composite fermion picture was successfully applied [24] to the independent dynamics of the compressible fluid at $\nu = 1/2$. This strongly-interacting, gapless ground state can be described as a Fermi liquid of composite fermions with vanishing effective magnetic field. Experiments [25] have confirmed this theory by observing the free motion of the composite fermions.

The chiral boson theory of the edge excitations

After the original works of Halperin [26] and Stone [16], a general theory of edge excitations, corresponding to the hierarchical constructions of wave-functions, has been formulated [27][15][28]. This is the $(1 + 1)$-dimensional theory of the chiral boson [29]. An equivalent description is given by Abelian Chern-Simons theories on $(2 + 1)$-dimensional open domains [28]. The edge excitations of the Laughlin fluid are described by a one-component chiral boson, while the hierarchical fluids require many components. Every boson describes an independent edge current, and thus the

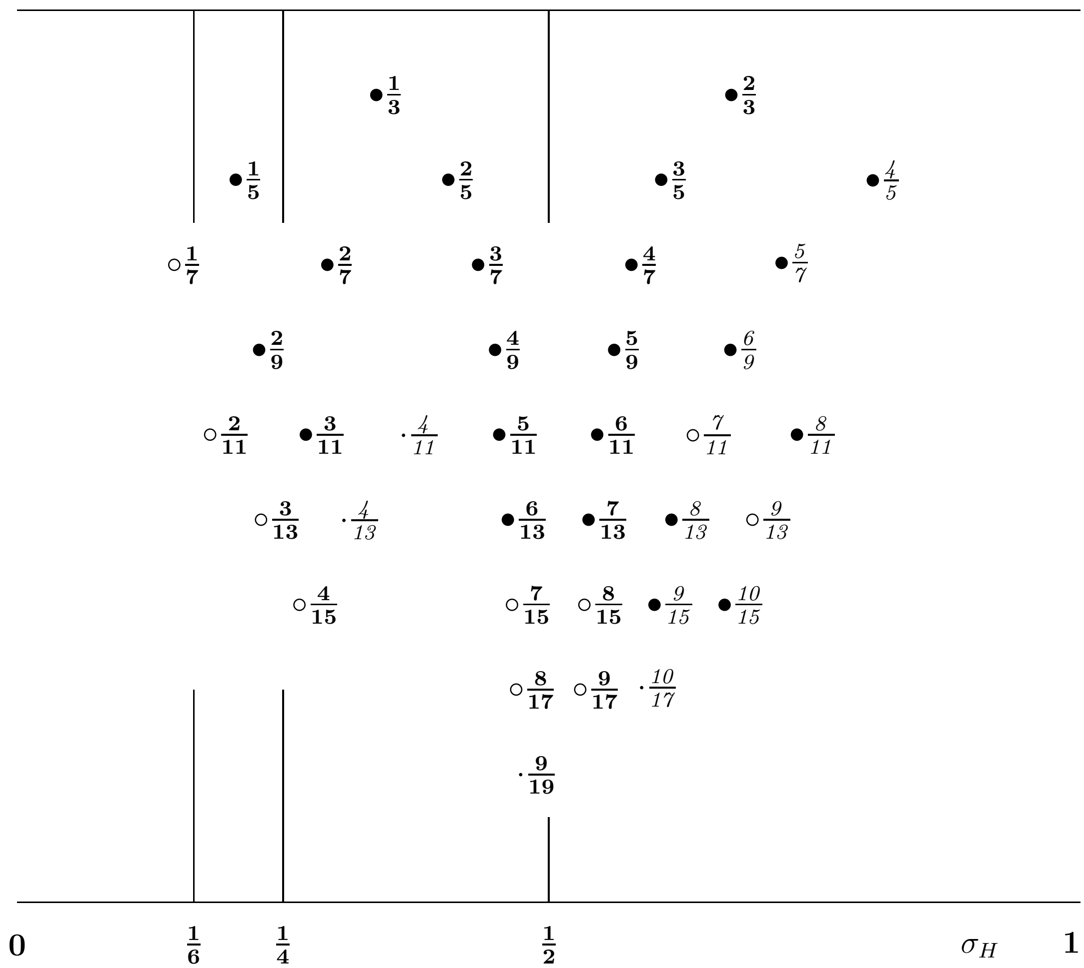

Figure 1: Experimentally observed plateaus in the range $0 < \nu < 1$: their Hall conductivity $\sigma_H = (e^2/h)\nu$ is displayed in units of (e^2/h). The points denote stability: $(\bullet)$ very stable, $(\circ)$ stable, and $(\cdot)$ less stable plateaus. Theoretically understood plateaus are in **bold**, unexplained ones are in *italic*. Observed cases of coexisting fluids are displayed as $\nu = 2/3, 6/9, 10/15$, $\nu = 3/5, 9/15$ and $\nu = 5/7, 15/21$ (but 15/21 is not displayed). (Adapted from ref. [31])

incompressible fluids have generically a composite edge structure. Each current gives rise to the Abelian current algebra (2.13), which implies the Virasoro algebra (2.14) with central charge $c = 1$ [3].

On an annulus geometry, with edge circles $|\mathbf{x}| = R_1$ and $|\mathbf{x}| = R_2$, one introduces m independent one-dimensional *chiral* currents

$$J^i\left(R_1\theta - v_i t\right) = -\frac{1}{2\pi R_1}\frac{\partial}{\partial\theta}\ \phi^i\ , \qquad (|\mathbf{x}| = R_1), \tag{3.1}$$

and corresponding ones with opposite chirality $J^i\left(R_2\theta + v_i t\right)$ at the other edge $|\mathbf{x}| = R_2$. The dynamics of these currents on the edge circle $|\mathrm{x}| = R_1$ is governed by the action,

$$S = -\frac{1}{4\pi}\int dt\ dx\ \sum_{i=1}^{m}\ \kappa_i\left(\partial_t\phi^i + v_i\partial_x\phi^i\right)\partial_x\phi^i\ , \qquad \text{for} \quad x \equiv R_1\theta\ , \tag{3.2}$$

for the m $(1+1)$-dimensional *chiral boson* fields ϕ^i [29]. The corresponding action for the other circle $x \equiv R_2\theta$ is obtained by replacing $v_i \to (-v_i)$. The dynamics on the two edges are identical and independent, only constrained by the conservation of the total charge: thus we describe one of them only. We can change the normalization of the fields, and reduce each coupling constant to a sign, $\kappa_i \to \pm 1$. The equations of motion imply that the fields are chiral, $\phi^i = \phi^i(x - v_i t)$, and canonical quantization implies the following commutation relations for the currents,

$$\left[J^i(x_1), J^k(x_2)\right] = \frac{1}{2\pi\kappa_i}\ \delta^{ik}\ \delta'(x_1 - x_2)\ , \qquad (t_1 = t_2)\ , \tag{3.3}$$

which are those of the multi-component Abelian current algebra $\widehat{U(1)}^{\otimes m}$ [3]. The positive definiteness of the Hamiltonian requires the signs of the velocities v_i and the couplings κ_i to be related: $v_i\kappa_i > 0\ , \qquad i = 1, \ldots, m$.

Let us discuss one particular chiral current, $v_i > 0$ (i.e. $\kappa_i = 1$). The quantization of the chiral boson is equivalent to the construction of the representations of the current algebra (3.3). Actually, all the states in the Hilbert space of the theory can be fitted into a set of representations [3]. To this end, we introduce the Fourier modes of the currents,

$$J^i(R\theta - v_i t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\ \alpha^i_n\ \mathrm{e}^{in(\theta - v_i t)}\ , \tag{3.4}$$

which satisfy,

$$\left[\alpha^i_n, \alpha^j_m\right] = \ \delta^{ij}\ \frac{n}{\kappa_i}\ \delta_{n+m,0}\ . \tag{3.5}$$

The positivity of the ground-state expectation value $\langle\Omega|\alpha^i_n\alpha^i_{-n}|\Omega\rangle \equiv \|\alpha^i_n|\Omega\rangle\|^2 \;\geq 0$, and the commutation relations (3.5) with $\kappa_i = 1$ imply the conditions

$$\alpha^i_n|\Omega\rangle = 0 \ , \qquad n > 0 \qquad\quad (v_i > 0). \tag{3.6}$$

An irreducible highest-weight representation of the $\widehat{U(1)}$ current algebra is made by the highest-weight state $|\Omega\rangle$ and by all states obtained by applying any number of $\alpha^i_n, \;\; n < 0$, to it. The weight of the representation is given by the eigenvalue of α^i_0, which is the single-edge charge, in units to be specified later. For the ground state, we have

$$\alpha^i_0|\Omega\rangle = 0 \ . \tag{3.7}$$

Other unitary highest-weight representations can be similarly built on top of other highest-weight states $|r\rangle$, $r \in \mathrm{R}$, which satisfy

$$\alpha^i_n|r\rangle = 0 \quad n > 0 \ , \qquad \alpha^i_0|r\rangle = r|r\rangle \ , \tag{3.8}$$

and are built by applying the *vertex operators* to the ground state [3]. These representations correspond to the quasi-particle excitations of this edge theory. The Virasoro generators are defined by the Sugawara construction [3],

$$L^i_n = \frac{\kappa_i}{2} \sum_{l=-\infty}^{\infty} \; : \; \alpha^i_{n-l}\alpha^i_l \; : \; . \tag{3.9}$$

They give rise to the Virasoro algebra (2.14) with $c = 1$, for each current component i.

The m-edge theory has $\widehat{U(1)}^{\otimes m}$ symmetry, $c = m$, and is parametrized by an integer, symmetric $(m \times m)$ matrix, with odd diagonal elements, the K matrix [28], which determines the Hall conductivity and the fractional charge, spin and statistics of the edge excitations. In the chiral boson theory, the fusion rules are the addition of weight vectors $\vec{r}$; the set of representations which is closed under these rules is the *lattice* Γ,

$$\Gamma = \left\{ \vec{r} \mid \; \vec{r} = \sum_{i=1}^{m} n_i \vec{v}_i \ , \quad n_i \in \mathbf{Z} \right\} \ . \tag{3.10}$$

The basis vectors $\vec{v}_i$ represent a physical elementary excitation in the i-th edge component, which may not correspond to the previous basis of propagating modes. The *physical charge* of an excitation with labels $n_i \in \mathrm{Z}$ is thus given by the sum of the components in the physical basis [11],

$$Q = \sum_{i,j=1}^{m} \; K^{-1}_{ij} \; n_j \ , \tag{3.11}$$

where

$$K^{-1}_{ij} = \sum_{l=1}^{m} \Lambda_{il} \frac{1}{\kappa_l} \Lambda^T_{lj} = (\vec{v}_i \cdot \eta \cdot \vec{v}_j) \ . \tag{3.12}$$

Similarly, the fractional spin and statistics of this excitation is given by the sum of the eigenvalues of the Virasoro generators L^i_0,

$$\frac{\theta}{\pi} = \sum_{i,j=1}^{m} n_i \ K^{-1}_{ij} \ n_j \ , \qquad n_i \in \mathbf{Z} \ . \tag{3.13}$$

In general, the metric K^{-1} of the lattice Γ in the basis $\vec{v}_i$ is pseudo-Euclidean with signature $\eta_{ij} = \delta_{ij}\kappa_i$, due to the possible presence of excitations with different chiralities.

The Hall conductivity in the annulus geometry can be measured by applying a uniform electric field along all the edges, $E^i = E$. The chiral anomaly of the edge theory actually corresponds to a radial flow of particles in the annulus, which move from the inner edge to the outer edge. The Hall conductivity can be thus found to be [6],

$$\sigma_H = \frac{e^2}{h} \ \nu \ , \qquad \nu = \sum_{i,j=1}^{m} \ K^{-1}_{ij} \ . \tag{3.14}$$

Equations (3.11-3.14) for the Hall conductivity and the spectrum of the charge and fractional statistics of edge excitations are the basic data of the quantum incompressible fluid described by the chiral boson theories [15][28]. The existence of m electron excitations with unit charge and integer statistic relative to all excitations, requires that K has integer entries with odd integers on the diagonal [28].

The Jain hierarchy

The Jain fluids have been described by the subset of the chiral boson theories characterized by the following K matrices [28],

$$K_{ij} = \pm\delta_{ij} + p \ C_{ij} \ , \qquad C_{ij} = 1 \ \forall \ i,j = 1,\ldots,m \ , \ p > 0 \text{ even} \ , \tag{3.15}$$

and the following spectra of edge excitations,

$$\begin{aligned} \nu &= \frac{m}{mp \pm 1} \ , \qquad p > 0 \text{ even} \ , \qquad (c = m) \ , \\ Q &= \frac{1}{pm \pm 1} \ \sum_{i=1}^{m} n_i \ , \\ \frac{\theta}{\pi} &= \pm \left(\sum_{i=1}^{m} n_i^2 - \frac{p}{mp \pm 1} \left(\sum_{i=1}^{m} n_i \right)^2 \right) \ . \end{aligned} \tag{3.16}$$

Note that K has $(m-1)$ degenerate eigenvalues $\lambda_i = 1,\ \ i = 1, \ldots, m-1$ (resp. $\lambda_i = -1$), and a single value $\lambda_m = \pm 1 + mp$. If the sign $\pm$ is negative, one edge has opposite chirality to the others. There is one basic charged quasi-particle excitation with label $n_i = (1, \ldots, 1)$ and $m(m-1)/2$ *neutral* excitations for $n_i = (\delta_{ik} - \delta_{il})$, $1 \leq k < l \leq m$, with identical integer statistics.

The corresponding trial wave functions for the ground state have been constructed by Jain as [23]

$$\Psi_\nu = D^{p/2}\ L^m\ \mathbf{1}\ , \qquad p \text{ even}\ , \tag{3.17}$$

where $L^m\ \mathbf{1}$ represents schematically the wave function of m filled Landau levels and $D^{p/2}$ multiplies the wave function by the p-th power of the Vandermonde determinant, which "attaches p flux tubes to each electron", and transforms them into "composite fermions".

The Jain hierarchy covers most of the experimentally observed plateaus, as we discuss in the next section. However, within the chiral boson approach, there is no clear motivation for selecting the special K matrices (3.15). The size of the gap for bulk density waves is usually invoked for solving this puzzle: the observed fluids are supposed to have the largest gaps, while the general K fluids have small gaps and are destroyed by thermal fluctuations and other effects. It is also true that edge theories give kinematically possible incompressible fluids and their universal properties, but cannot describe the size of the gaps, which is determined by the microscopic bulk dynamics.

Nevertheless, we have found a natural way to select the Jain hierarchy within the $W_{1+\infty}$ edge theory approach [11]. Indeed, the Jain fluids correspond to the $W_{1+\infty}$ minimal models, which are characterized by possessing less states than their chiral boson counterparts. We propose this reduction of available states as a natural stability principle.

Experiments

We first discuss the spectrum of fractional Hall conductivities in eq. (3.16). According to Jain, the stability of the ground states (3.17) should be approximately independent of m, which counts the number of Landau levels filled by the composite fermion. This is in analogy with the integer Hall plateaus, which are all equally stable. On the other hand, the stability decreases by increasing $|p|$, as observed for the Laughlin fluids ($m = 1$). Therefore, the most stable family of plateaus is,

$$\nu = \frac{m}{2m \pm 1}\ , \qquad (p = 2)\ , \tag{3.18}$$

which accumulate at $\nu = 1/2$. The next less stable family is

$$\nu = \frac{m}{4m \pm 1} , \qquad (p = 4) , \tag{3.19}$$

which accumulate at $\nu = 1/4$. This behaviour is clearly seen in the experimental data of fig. 1. For these filling fractions, the Jain wave functions for the ground state and the simplest excited states have a good overlap with those obtained numerically by diagonalizing the microscopic Hamiltonian with a small number of electrons.

A closer look into the experimental values of the filling fractions in fig. 1 shows other points (in italic), like $\nu = 4/5,\ 5/7,\ 8/11$, which fall outside the Jain main series (3.18,3.19) (in bold). These points were originally interpreted as "charge conjugates" of these series [23],

$$\nu = 1 - \frac{m}{2m \pm 1} , \qquad \nu = 1 - \frac{m}{4m \pm 1} , \tag{3.20}$$

which actually belong to the second iteration of the Jain hierarchy [23]. Unfortunately, the charge conjugate states do not fit well the data in fig. 1. The $\nu = 1/2$ family would be self-conjugate; thus there should be two fluids per filling fraction, which are not observed, apart from two cases. Actually, coexisting fluids can be detected by experiments where the magnetic field is tilted from the orthogonal direction to the plane [30]. Furthermore, the conjugate of the observed fractions in the $\nu = 1/4$ family are not observed in half of the cases. Finally, there are fractions which do not belong to any previous group: $\nu = 4/11,\ 7/11,\ 4/13,\ 8/13,\ 9/13,\ 10/17$.

In conclusion, all the fractions outside the main Jain families ((3.18,(3.19) are not well understood at present (and will not be explained in this paper). Any known extension of the previous theory which explains these few extra fractions, also introduces many more unobserved fractions, with an unclear pattern of stability. Besides the second iteration of the Jain hierarchy [23], we also quote the approach proposed by Fröhlich and collaborators [31]. They analyzed all lattices Γ (3.10), with positive-definite, integer (inverse) metric K, for small values of $det(K)$, whose classification is known in the mathematical literature. These lattices can be related to the $SU(m)$, $SO(k)$ and exceptional Lie algebras. The stability of the corresponding fluids does not follow a clear pattern related to these algebras, besides the case of the chiral Jain fluids (3.15), whose $SU(m)$ symmetry [32] will explained in the next section. Moreover, in this approach, the K matrices for the Jain filling fractions $\nu = m/(mp-1)\ , p > 0$, are different from the Jain proposal (3.15) which is not positive definite.

In figure 2, we study the limitations of phenomenological descriptions of the stability of the fluids. Besides all the observed (bold) fractions of fig. 1, we report the

1/3	**2/3**								σ_H
1/5	**2/5**	**3/5**	**4/5**						
1/7	**2/7**	**3/7**	**4/7**	**5/7**					
	2/9	*3/9*	**4/9**	**5/9**	**6/9**	*7/9*			
	2/11	**3/11**	**4/11**	**5/11**	**6/11**	**7/11**	**8/11**		
		3/13	**4/13**	*5/13*	**6/13**	**7/13**	**8/13**	**9/13**	*10/13*
		3/15	**4/15**	*5/15*	*6/15*	**7/15**	**8/15**	**9/15**	**10/15**
			4/17	*5/17*	*6/17*	*7/17*	**8/17**	**9/17**	**10/17**
								9/19	

Figure 2: List of all fractions $\nu = p/q$, with $2/11 < \nu < 4/5$, $1 < p \leq 10$ and $3 \leq q \leq 17$, q odd. The fractions corresponding to experimental values of the Hall conductivity $\sigma_H = (e^2/h)\nu$ are in **bold**; the unobserved fractions are in *italic*. Observed fractions joined by lines are explained by the Jain hierarchy (3.18,3.19).

unobserved (italic) ones $\nu = p/q$, which satisfy the conservative cuts of the "phase space"

$$\frac{2}{11} < \nu = \frac{p}{q} < \frac{4}{5} \ , \qquad \text{and} \ \ p \leq 10 \ , q \leq 17 \ . \tag{3.21}$$

Namely, we display all fractions which would be observed if the gap were a smooth function of the parameters (ν, p, q) interpolating the data, a typical phenomenological hypothesis. Figure 2 shows that, besides the families ((3.18,(3.19), about half of the fractions are unexplained observed fillings and half are unobserved but phenomenologically possible. This implies that the gap is not a smooth function of simple parameters like (ν, p, q) - deeper theories are needed to explain stability.

Actually, a major virtue of the Jain hierarchy is that of representing one-parameter families of Hall states, within which the gap *is* a smooth function of the above parameters. We think of these families as the set of *kinematically allowed* quantum incompressible fluids (at first level of the hierarchy).

More specific confirmations of the edge theory (3.15) come from the experimental tests of the spectrum of excitations (3.16). An experiment with high time resolution [33] has measured the propagation of a single chiral charge excitation for $\nu = 1/3$ $(m = 1 \ , p = 2)$, and $\nu = 2/3$ $(m = 2, \ p = 2)$; this is in agreement with the Jain theory, although the neutral excitations have not been seen yet. The resonant tunnelling experiment [34] has verified the conformal dimensions (3.16) for the simplest Laughlin fluid $\nu = 1/3$ [35]. Extensions of this experiment to $(m > 1)$ fluids have been suggested, as well as tests of the neutral edge spectrum [36]. We shall discuss more these experiments in section 4.

4 $W_{1+\infty}$ minimal models

The theory of $W_{1+\infty}$ representations

We now develop the classification program of incompressible quantum Hall fluids outlined in section two. The basic piece of information we need is the mathematical theory of $W_{1+\infty}$ representations . Luckily enough, all unitary, irreducible (quasi-finite) $W_{1+\infty}$ representations were obtained in the fundamental work by Kac and collaborators [19][20]: they exist for integer central charge $(c = m)$ and can be regular, i.e. *generic*, or *degenerate*. In ref.[9], we used the generic representations to build the *generic* $W_{1+\infty}$ theories, which were shown to correspond to the previously described m-component chiral boson theories parametrized by generic $(m \times m)$ K matrices. In the algebraic approach, this is proven by identifying the generic $W_{1+\infty}$

representations with $\widehat{U(1)}^{\otimes m}$ representations . Both representations are labelled by the same weight vectors $\vec{r}$. A complete equivalence requires also a one-to-one map between the states built on top of the respective highest weight states. The general theory of $W_{1+\infty}$ representations [19][20], leads to the following relations between irreducible representations of the two algebras,

$$\begin{aligned}
M\left(W_{1+\infty}\ ,1,r\right) &\sim M\left(\widehat{U(1)},1,r\right)\ ; \\
M\left(W_{1+\infty}\ ,m>1,\vec{r}\right) &\sim M\left(\widehat{U(1)}^{\otimes m},m,\vec{r}\right)\ , \qquad \text{for } (r_i-r_j)\notin \mathbf{Z}\ ,\forall\ i\neq j\ , \\
M\left(W_{1+\infty}\ ,m>1,\vec{r}\right) &\subset M\left(\widehat{U(1)}^{\otimes m},m,\vec{r}\right)\ , \qquad \text{if } \exists\ (r_i-r_j)\in \mathbf{Z}\ .
\end{aligned} \tag{4.1}$$

Generically, $W_{1+\infty}$ and $\widehat{U(1)}^{\otimes m}$ representations are one-to-one equivalent. The exceptions appear for $c>1$, when the weight has some integer components $(r_i - r_j)$. In these cases, the relation is many-to-one, i.e. an irreducible $\widehat{U(1)}^{\otimes m}$ representation is *reducible* with respect to the $W_{1+\infty}$ algebra. We call *generic* the $W_{1+\infty}$ representations which are one-to-one equivalent to $\widehat{U(1)}^{\otimes m}$ ones ($(r_i - r_j) \notin \mathbf{Z}\ ,\forall\ i \neq j$), and *degenerate* the remaining $W_{1+\infty}$ representations ($\exists\ (r_i - r_j) \in \mathbf{Z}$).

The results (4.1) allow the construction of several types of $W_{1+\infty}$ symmetric theories. The generic $W_{1+\infty}$ theories [9] are defined by lattices Γ (3.10) which contain generic $W_{1+\infty}$ representations only: for these, the basis vectors satisfy $((\vec{v}_\alpha)_i - (\vec{v}_\alpha)_j) \notin \mathbf{Q}$, $\forall\ \alpha, i \neq j = 1,\ldots,m$. These theories are thus equivalent to chiral boson theories[‖]. Other $W_{1+\infty}$ theories, containing only degenerate representations , are instead different. These are the *minimal models*, which we shall describe below. They are the basic new $W_{1+\infty}$ theories, and are actually very important, because they will be shown to correspond to the experimentally observed Jain fluids. The mathematical rules for building the degenerate representations have a hierarchical structure similar to the Jain construction: in ref. [11], we fully explained this correspondence to the lowest order of the hierarchies.

On the other hand, the chiral boson theories of the Jain hierarchy (3.16) have been widely used in the literature and partially confirmed by the experiments, as we discuss hereafter. These are also $W_{1+\infty}$ symmetric, but are not the simplest realizations of this symmetry, because their $\widehat{U(1)}^{\otimes m}$ representations are reducible. Reducible and irreducible degenerate representations have the same quantum numbers of fractional charge, spin and statistics. The existing experiments at hierarchical filling fractions were sensible to these data only; therefore their successful interpretation within the

‖ The ground state representation ($\vec{r} = 0$) must also be a $\widehat{U(1)}^{\otimes m}$ representation for the closure of the fusion rules.

chiral-boson theory is consistent with our theory. More refined experiments are needed to test the difference.

The $W_{1+\infty}$ Minimal Models

Degenerate representations are common in conformal field theory: if the central charge and the weight of a given representation satisfy certain algebraic relations, some of its states decouple, and should be projected out to obtain an irreducible representation. A general fact is that the fusion of degenerate representations only gives degenerate representations of the same type; thus it is possible to find sets of degenerate representations which are closed under the fusion rules; these build the minimal models [3]. There are specific minimal models for any symmetry algebra: the well-known ones are the $c < 1$ Virasoro minimal models; larger symmetry algebras, like $W_{1+\infty}$, have $c > 1$ minimal models [3]. The minimal models have less states than the generic theories with the same symmetry, due to the projection; for the same reason, they have a richer dynamics.

The $W_{1+\infty}$ minimal models are *not* realised by the multi-component chiral boson theories with $\widehat{U(1)}^{\otimes m}$ symmetry, because the latter do not incorporate the projection for making irreducible the $W_{1+\infty}$ representations of degenerate type. They are instead realised by the $\widehat{U(1)} \otimes \mathcal{W}_m(p=\infty)$ conformal theories [20], where the $\mathcal{W}_m(p)$ are the Zamolodchikov-Fateev-Lykyanov models with $c = (m-1)\left[1 - m(m+1)/p(p+1)\right]$ [37]. We have found the minimal set of representations which are closed under the fusion rules of these models, which is again a lattice (3.10) satisfying special conditions, which makes it similar to the weight lattice of the $SU(m) \otimes U(1)$ Lie algebra [11]. We have also identified the physical charge of the excitations and the Hall conductivity with arguments analogous to the one described before in the chiral boson theory. We have obtained the spectrum

$$\begin{aligned}
\nu &= \frac{m}{mp \pm 1} , \qquad p > 0 \text{ even} , \qquad c = m , \\
Q &= \frac{1}{pm \pm 1} \sum_{i=1}^{m} n_i , \qquad n_1 \geq n_2 \geq \cdots \geq n_m , \quad n_i \in \mathbf{Z} , \\
\frac{\theta}{\pi} &= \pm \left(\sum_{i=1}^{m} n_i^2 - \frac{p}{mp \pm 1} \left(\sum_{i=1}^{m} n_i \right)^2 \right) .
\end{aligned} \tag{4.2}$$

These spectra agree with the experimental data and match the results of the lowest-order Jain hierarchy (3.16)** discussed in section 3.

** Note, however, the reduced multiplicities of eq.((4.2).

This result has far-reaching consequences, both theoretical and experimental. The physical mechanism which stabilizes the observed quantum Hall fluids has both short and long distance manifestations. At the microscopic level, it can be described by the Jain composite-electron picture and by the size of the gaps; in the scaling limit, by the minimality of the $W_{1+\infty}$ edge theory. Actually, we find it rather natural that the theories with a minimal set of excitations are also dynamically more stable. This long-distance stability principle leads to a logically self-contained edge theory of the fractional Hall effect: a thorough derivation of experimental results is obtained from the principle of $W_{1+\infty}$ symmetry, which is the basic property of the Laughlin incompressible fluid. This independent hierarchical construction is the main result of our approach.

Furthermore, the detailed predictions of the $W_{1+\infty}$ minimal theories are different from those of the chiral boson theories. The main differences are as follows:

i) There is a *single* Abelian current, instead of m independent ones, and therefore a single elementary (fractionally) charged excitation; there are neutral excitations, but they cannot be associated to $(m-1)$ independent edges.

ii) The dynamics of these neutral excitations is new: they have associated an $SU(m)$ (not $\widehat{SU(m)}_1$) "isospin" quantum number, which is given by the highest weight [11],

$$\mathbf{\Lambda} = \sum_{a=1}^{m-1} \mathbf{\Lambda}^{(a)} \ (n_a - n_{a+1}) \ , \tag{4.3}$$

where $\mathbf{\Lambda}^{(a)}$ are the fundamental weights of $SU(m)$ [38] and $\{n_i\}$ are the integer labels of eq.(4.2). More precisely, they are associated to $\mathcal{W}_m(p=\infty)$ representations, whose fusion rules are isomorphic to the branching rules of the $SU(m)$ Lie algebra. Therefore, the neutral excitations are quark-like and their statistics is non-Abelian. For example, the edge excitation corresponding to the electron is composed of them, and carries both the additive electric charge and the $SU(m)$ isospin.

iii) The degeneracy of particle-hole excitations at fixed angular momentum is modified by the projection of the minimal models. This counting of states is provided by the characters of degenerate $W_{1+\infty}$ representations, which are known [20]. If the neutral $SU(m)$ excitations have a bulk gap, the particle-hole degeneracy of the ground state (the Wen topological order on the disk [39]) is different from the corresponding one of $\widehat{U(1)}^{\otimes m}$ excitations. This can be tested in numerical diagonalizations of few electron systems.

Non-Abelian fusion rules and non-Abelian statistics

The physical electron is identified as the minimal set of $W_{1+\infty}$ representations with unit charge and integer statistics relative to all excitations in eq.(4.2) [11]. These conditions are fulfilled by a composite edge excitation $n_i = (1+p, p, \ldots, p)$, which is made of (mp) elementary charged *anyons* and the *quark* elementary neutral excitation, i.e. the fundamental $SU(m)$ isospin representation , due to $(n_i - n_{i+1}) = \delta_{i,1}$ in (4.3), for filling fraction $\nu = m/(mp \pm 1)$.

A conduction experiment which could show the composite nature of the electron has been proposed [36]. It is a modification of the "time-domain" experiment [33], in which a very fast electric pulse was injected at the boundary of a disk sample and a chiral wave was detected at another boundary point. The proposed experiment will also detect the neutral excitation in the electron, which propagates at a different speed.

The compositeness of the electron also plays a role in the resonant tunnelling experiment [34], in which two edges of the sample are pinched at one point, such that the corresponding edge excitations, having opposite chiralities, can interact. At $\nu = 1/3$, the point interaction of two elementary anyons is relevant and determines the *scaling law* $T^{2/3}$ for the conductance [35]. This scaling of the tunnelling resonance peaks is verified experimentally. On the other hand, off-resonance and at low temperature, the conductance is given by the tunnelling of the whole electron, with a different scaling law in temperature [15].

These experiments involve processes with one or two electrons: their quark compositeness can be seen in four-electron processes, like scattering. Indeed, the expansion of the four-point function of the electrons in intermediate channels is determined by the fusion ($SU(2)$ isospin for $m = 2$) of an electron pair. This is, schematically,

$$\langle\Omega|\Psi^\dagger(1)\Psi^\dagger(2)\Psi(3)\Psi(4)|\Omega\rangle = \sum_{s=0,1} \langle\Omega|\Psi^\dagger(1)\Psi^\dagger(2)|\{s\}\rangle\langle\{s\}|\Psi(3)\Psi(4)|\Omega\rangle \ , \qquad (4.4)$$

where the two channels follow from the addition of two one-half isospin values. More than one intermediate channel are also created in the adiabatic transport of two electrons around each other, in presence of two other excitations, because the amplitude for this process is again a four-point function. For generic excitations, the monodromy phases form a matrix, which gives a non-Abelian representation of the braid group [17]. This is precisely the notion of non-Abelian statistics††. These monodromy properties also determine the degeneracy of the ground state on a torus geometry,

†† For a general discussion of non-Abelian statistics in the quantum Hall effect, see ref.[40].

the so-called topological order [15]. This depends on the type of the representations carried by the edge excitations [3], and has not yet been computed for the $\widehat{U(1)} \otimes \mathcal{W}_m$ ones.

The degeneracy of excitations above the ground state

In order to discuss this point, we must rewrite the spectrum (4.2). Let us consider, for example, the $m = 2$ chiral theories, which are relevant for $\nu = 2/5, \ldots$. The corresponding $W_{1+\infty}$ minimal model is constructed from degenerate representations of the type $\widehat{U(1)} \otimes \mathcal{W}_2$, where the $\mathcal{W}_2$ algebra is the $c = 1$ Virasoro algebra. As explained in [11], these degenerate Virasoro representations carry an $SU(2)$ isospin quantum number, as required by the fusion rules [3]. Consider any excitation (n_1, n_2) associated to a $\widehat{U(1)} \otimes \mathrm{Vir}$ representation , labelled by the $\widehat{U(1)}$ charge $Q \propto (n_1 + n_2)$ and the $SU(2)$ isospin $s = |n_1 - n_2|/2$. Divide the square lattice (n_1, n_2) into charged excitations and their neutral daughter excitations by introducing the change of integer variables $(n_1, n_2) \to (l, n)$:

$$\mathrm{I}: \begin{cases} 2l & = n_1 + n_2 \\ 2n & = n_1 - n_2 > 0 \\ & \quad (n_1 + n_2 \text{ even}), \end{cases} \qquad \mathrm{II}: \begin{cases} 2l+1 & = n_1 + n_2 \\ 2n+1 & = n_1 - n_2 > 0 \\ & \quad (n_1 + n_2 \text{ odd}). \end{cases} \tag{4.5}$$

The spectrum (4.2) can be rewritten, for $\nu = 2/(2p+1)$,

$$\mathrm{I}: \begin{cases} Q & = \frac{2l}{2p+1} \ , \\ \frac{1}{2}\frac{\theta}{\pi} & = \frac{1}{2p+1} l^2 + n^2 \end{cases} \qquad \mathrm{II}: \begin{cases} Q & = \frac{2}{2p+1}\left(l + \frac{1}{2}\right) \\ \frac{1}{2}\frac{\theta}{\pi} & = \frac{1}{2p+1}\left(l + \frac{1}{2}\right)^2 + \frac{(2n+1)^2}{4} \end{cases} . \tag{4.6}$$

The $\widehat{U(1)}$ charged excitations have the same spectrum $Q = \nu k$, $\theta/\pi = \nu k^2$, of the simpler Laughlin fluids ($m = 1$). Moreover, the infinite tower of neutral daughters ($n > 0$) are characterized by the conformal dimensions $h = (2n)^2/4$ (resp. $h = (2n+1)^2/4$).

The number of excitations above the ground state depends on whether the neutral excitations have a bulk gap or not. This affects also the thermodynamic quantities like the specific heat.

As said before, the charged edge excitations correspond to Laughlin quasi-particles vortices in the bulk of the incompressible fluid, which spill their density excess or defect to the boundary. They have an (non-universal) gap proportional to the electrostatic energy of the vortex core, which is not accounted for by the edge theory [2]. On the other hand, the bulk excitations corresponding to *neutral* edge excitations are not well understood yet. If they have a gap, they could exhibit the internal structure

of the quasi-particle vortex, or be bound states of a quasi-particle and a quasi-hole; these would be localized two-dimensional excitations. Neutral and charged gapful excitations can be thought of as analogs of the *breathers* and *solitons* of one-dimensional integrable models, respectively, [41]. On the other hand, gapless neutral excitations would be pure effects of the structured edge.

In the gapful case, the excitations above the ground state are the particle-hole excitations (2.15), which are described by the $W_{1+\infty}$ representation $(n = l = 0)$ in (4.6). The degeneracy of these states is encoded in the character of the representation [3]. In the gapless case, the states contained in the neutral daughter Virasoro representations $(n > 0\ ,\ l = 0)$ also contribute to these degeneracies, because they have integer spin (Virasoro dimension) and are indistinguishable. Actually, the infinite tower of Virasoro representations $(n > 0\ ,\ l$ fixed) of each charged parent state $(l, n = 0)$ can be summed (with their multiplicity one) into a single $\widehat{U(1)}$ representation [11]. In this case, the $m = 2$ $W_{1+\infty}$ square-lattice spectrum (4.6) reduces to a one-dimensional array of $\widehat{U(1)}^{\otimes 2}$ representations , with spectrum

$$\mathrm{I}:\ \frac{1}{2}\frac{\theta}{\pi} = \frac{1}{2p+1} l^2\ , \qquad\qquad \mathrm{II}:\ \frac{1}{2}\frac{\theta}{\pi} = \frac{1}{2p+1}\left(l + \frac{1}{2}\right)^2 + \frac{1}{4}\ , \tag{4.7}$$

where the second $\widehat{U(1)}$ eigenvalue is not observable.

We can repeat this resummation for the corresponding chiral boson theory of the Jain fluid [11]. The spectrum of charge and fractional statistics is again given by (4.6), with multiplicities given by $n \in$ Z: each (l, n) value corresponds to a $\widehat{U(1)} \otimes \widehat{U(1)}$ representation now. If they are gapless, the neutral daughter $\widehat{U(1)} \otimes \widehat{U(1)}$ representations $((l, n),\ n \neq 0 \in \mathbf{Z})$ of each charged representation $(l, 0)$ can be similarly summed up into one representation of the larger algebra $\widehat{U(1)} \otimes \widehat{SU(2)}_1$, the non-Abelian current algebra of level one [3][28]. The spectrum of $\widehat{U(1)} \otimes \widehat{SU(2)}_1$ representations is again given by (4.7).

We can now compare the predictions of the $W_{1+\infty}$ minimal models and the chiral boson theories for the degeneracy of the excitations above the ground state. This degeneracy can be measured in numerical simulations of a few electron system in the disk geometry, by charting the eigenstates of the Hamiltonian below the bulk gap [15][39]. Consider, for example, the $\nu = 2/5$ $(m = p = 2)$ ground state $((l = 0, n = 0)$ in ((4.6)). In the following table, we report the degeneracies encoded in the characters

of the relevant $\widehat{U(1)} \otimes \mathrm{Vir}$, $\widehat{U(1)}^{\otimes 2}$ and $\widehat{U(1)} \otimes \widehat{SU(2)}_1$ representations [11]:

ΔJ	0	1	2	3	4	5
$\widehat{U(1)} \otimes \mathrm{Vir}$	1	1	3	5	10	16
$\widehat{U(1)} \otimes \widehat{U(1)}$	1	2	5	10	20	36
$\widehat{U(1)} \otimes \widehat{SU(2)}_1$	1	4	9	20	42	80

(4.8)

As said before, if neutral daughter excitations are gapful, they should not be counted, and the degeneracy is only given by the particle-hole excitations encoded in the ground state character of each theory. On the other hand, if they are gapless, the total degeneracy is given by the characters of the resummed representations [11]. We conclude that:

i) The observation of $\widehat{U(1)} \otimes \mathrm{Vir}$ degeneracies confirms the $W_{1+\infty}$ minimal theory with gapful neutral excitations;

ii) The $\widehat{U(1)} \otimes \widehat{U(1)}$ degeneracies are found both in the $W_{1+\infty}$ minimal theory with gapless neutral excitations and in the chiral boson theory with gapful ones;

iii) The $\widehat{U(1)} \otimes \widehat{SU(2)}_1$ degeneracies support the chiral boson theory with gapless neutral excitations.

Numerical results known to us [15] are not accurate enough to see the differences in table ((4.8). Note the characteristic reduction of states of $W_{1+\infty}$ minimal models.

These remarks on the gap for neutral excitations do not affect the previous discussion of the conduction experiments, where excitations move along one edge or are transferred between two edges at the same Fermi energy, such that bulk excitations are never produced. Although the resummation of the neutral daughter $\mathcal{W}_m$ excitations gives Abelian excitations, these are not $W_{1+\infty}$ irreducible, and thus unlikely to be produced experimentally. We think that only irreducible $W_{1+\infty}$ excitations, i.e., the elementary ones, can be naturally produced in a real system by an external probe, for example by injecting an electron at the edge.

Remarks on the $SU(m)$ and $\widehat{SU(m)}_1$ symmetries

We would like to explain the type of non-Abelian symmetry of the $W_{1+\infty}$ minimal models and clarify the differences with the chiral boson theories of the Jain hierarchy, which have been also assigned the $SU(m)$ and $\widehat{SU(m)}_1$ symmetries [32][28][31][36].

Due to the $\widehat{U(1)} \otimes \mathcal{W}_m$ construction of the $W_{1+\infty}$ models, their excitations carry a quantum number which adds up as a $SU(m)$ isospin. This *does not* imply that these models have the full $SU(m)$ symmetry, in the usual sense of, say, the quark model of

strong interactions, because the states in each $\mathcal{W}_m$ representation do not form $SU(m)$ multiplets. As shown by the $m=2$ case, the quantum number $s=n/2$ of Virasoro representations is like the total isospin $S^2=s(s+1)$, but the S_z component is missing. In some sense, the effects of the $\mathcal{W}_m$ non-Abelian fusion rules can be thought of as a *hidden* $SU(m)$ symmetry.

On the other hand, it has been claimed that the chiral boson theories of the Jain hierarchy have a $SU(m)$ symmetry. The correct statement is, however, that they possess $\widehat{U(1)}\otimes\widehat{SU(m)}_1$ symmetry. This means that their $\widehat{U(1)}^{\otimes m}$ representations can be rearranged into representations of the $\widehat{U(1)}\otimes\widehat{SU(m)}_1$ current algebra. In the $\widehat{SU(m)}_k$ current algebra, the weights cannot be arbitrary, but are cut-off by the *level* k (e.g. for $m=2$, the spin s can be $0\le s\le k/2$) [3]. The level-one non-Abelian current algebra has very elementary representations and their fusion rules are made Abelian by this cut-off.

Therefore, the $\widehat{SU(m)}_1$ symmetry has no non-Abelian physical effect, it is only a convenient reorganization of the Abelian current algebra. The non-Abelian character of the excitations is a characteristic feature of the $W_{1+\infty}$ minimal models.

5 Further developments

In this paper, we have reviewed the $W_{1+\infty}$ theory of the edge excitations in the Quantum Hall Effect. In particular, we considered the *simplest* $W_{1+\infty}$ minimal models, which are made of *one-congruence-class* degenerate representations only [20]. It would be interesting to generalize this construction, in view of describing the experimentally observed filling fractions $4/5$, $5/7$, $4/11$, $7/11$, $8/11$, $4/13$, $8/13$, $9/13$, $10/17$, not explained here. Actually, the $W_{1+\infty}$ minimal models can be generalized by considering two (or more) congruence classes [11]. There are analogies between this mathematical construction and the Jain hierarchical construction of wave functions, which read

$$\Psi_\nu = D^{q/2}\ L^l\ D^{p/2}\ L^m\ \mathbf{1}, \qquad\qquad p,q \text{ even}, \tag{5.1}$$

to second order of iteration [23]. The number of fluids in any $W_{1+\infty}$ congruence class corresponds to the number of Landau levels in (5.1); in both constructions, there are two independent elementary anyons, each one accompanied by neutral excitations. However, we have not yet proven a complete equivalence of the two second-order hierarchies.

Another interesting development is the extension of the $W_{1+\infty}$ symmetry by addi-

tional degrees of freedom to the quantum incompressible fluid, which might describe spinful electrons or multi-layer Hall devices.

Acknowledgements

We would like to thank the organizers of the conference on *Statistical Mechanics and Quantum Field Theory* and the summer school on *Particles, Fields and Strings* for succeeding in creating a stimulating atmosphere for discussions and a positive interchange of ideas. Throughout our project, we have greatly benefited from discussions with V. Kac. We would also like to acknowledge the continuing support of L. Alvarez-Gaumé and S. Fubini. This work was supported in part by the CERN Theory Division, the MIT Center for Theoretical Physics and the European Community program "Human Capital and Mobility".

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